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Evaluation of $\mathrm{Tr}(J_\lambda^{2p})$ using the Brillouin function

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Abstract. We obtain expressions for $\mathrm{Tr}(J_\lambda^{2p})$ in terms of the Brillouin function. Standard properties of $\mathrm{Tr}(J_\lambda^{2p})$ are derived from them. Sum rules for the Bernoulli numbers and the Riemann zeta functions are deduced as corollaries.

1. Introduction

Expressions for $\mathrm{Tr}(J_\lambda^{2p})$ ($\lambda = x$ or y or z, $p \geqslant 0$) are available in the literature in terms of the Bernoulli polynomials (Ambler *et al* 1962, Subramanian and Devanathan 1974, De Meyer and Vanden Berghe 1978) and the hypergeometric functions (Rashid 1979, Ullah 1980), J_λ being the angular momentum matrices. Evaluation of $\mathrm{Tr}(J_\lambda^{2p})$ using the Brillouin function (Van Vleck 1932, Mattis 1965) could have been a natural corollary to some studies in magnetism, e.g. anisotropy constants of rare earth metals as functions of temperature and atomic number (Kazakov and Andreeva 1970). The purpose of this paper is to obtain (§ 2) expressions for $\mathrm{Tr}(J_\lambda^{2p})$ in terms of the Brillouin function $B_j(x)$, and derive from them (§ 3) the standard properties of $\mathrm{Tr}(J_\lambda^{2p})$ (Subramanian and Devanathan 1974, 1980, 1985, to be referred to as I, II and III respectively). As corollaries we obtain sum rules for the Bernoulli numbers and the Riemann zeta functions (§ 4).

2. Expressions for $\mathrm{Tr}(J_\lambda^{2p})$ in terms of $B_j(x)$

The starting point of our calculations is the partition function

$$Z = \sum_{m=-j}^{j} \exp(mx/j) \tag{1}$$

where $j > 0$ is the angular momentum quantum number (in units of $\hbar$). Since $mx = (-m)(-x)$, Z remains unaltered under the operation $x \to -x$. Hence Z is an even function of x. The average value of $(m/j)^p$, $p \geqslant 0$, is defined as

$$\langle (m/j)^p \rangle = Z^{-1} \mathrm{D}^p(Z) \qquad \mathrm{D} \equiv \mathrm{d}/\mathrm{d}x \qquad p \geqslant 0. \tag{2}$$

In this paper, we follow the convention that $\mathscr{H}^0 = 1$ for any operator $\mathscr{H}$. It may be noted that although $\langle m/j \rangle = \mathrm{D}(\ln Z)$, in general $\langle (m/j)^p \rangle \neq \mathrm{D}^p(\ln Z)$ and hence equation (16) of Kazakov and Andreeva (1970) needs a correction.

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2.1. Operator form for $Tr(J_\lambda^{2p})$

Since (see, for example, Van Vleck 1932)

$$Z^{-1}\mathrm{D}(Z) = \mathrm{D}(\ln Z) = B_j(x)$$
$$= [(2j+1)/2j] \coth((2j+1)x/2j) - (1/2j)\coth(x/2j) \tag{3}$$

and

$$\mathrm{D}^p(Z) = \mathrm{D}(Z\langle (m/j)^{p-1}\rangle) \qquad p \geqslant 1 \tag{4}$$

we have, in general,

$$\langle (m/j)^p \rangle = (B_j(x) + \mathrm{D})^p 1 \qquad p \geqslant 0 \tag{5}$$

so that

12014 $$\mathrm{Tr}(J_\lambda^p) = \sum_{m=-j}^{j} m^p = \lim_{x\to 0} Z\langle m^p \rangle = (2j+1)j^p \lim_{x\to 0} (B_j(x)+\mathrm{D})^p 1. \tag{6}$$

Since $B_j(x)$ is an odd function of x (see equation (3)), the operator $B_j(x)+\mathrm{D}$ has an odd parity. Hence the familiar result that $\mathrm{Tr}(J_\lambda^{2p+1}) = 0$ is immediately obtained. The non-trivial traces are given by

$$\mathrm{Tr}(J_\lambda^{2p}) = (2j+1)j^{2p} \lim_{x\to 0} (B_j(x)+\mathrm{D})^{2p} 1. \tag{7}$$

Equation (7) is a compact expression for $\mathrm{Tr}(J_\lambda^{2p})$ in terms of the Brillouin function. It reminds one of a similar relation for the Hermite polynomials $H_n(x)$ (Arfken 1970):

$$H_n(x) = (2x - \mathrm{D})^n 1 \qquad n \geqslant 0. \tag{8}$$

The operator expansion of $(B_j(x)+\mathrm{D})^{2p}$ can be done with the help of the Zassenhaus formula (Wilcox 1967, Witschel 1975)

$$\exp(\hat{P}+\hat{Q}) = \exp \hat{P} \exp \hat{Q} \exp \hat{C}_2 \exp \hat{C}_3 \ldots \exp \hat{C}_r \ldots \tag{9}$$

and the comparison method due to Witschel (1975). The disentangled and the undisentangled forms of equation (9) are expanded in terms of an ordering scalar parameter g and the operator coefficients of corresponding powers of g are compared:

$$\exp\{g(\hat{P}+\hat{Q})\} = \exp(g\hat{P}) \exp(g\hat{Q}) \exp(g^2\hat{C}_2) \exp(g^3\hat{C}_3) \ldots \tag{10}$$

$$\sum_{n=0}^{\infty} g^n(\hat{P}+\hat{Q})^n/n! = \sum_{s,t,u,v,\ldots=0}^{\infty} (g^{s+t+2u+3v+\ldots}/s!t!u!v!\ldots)\hat{P}^s\hat{Q}^t\hat{C}_2^u\hat{C}_3^v \ldots. \tag{11}$$

The operators C_r, obtained using the recurrence relations given by Wilcox (1967), for the special case of $P = B_j(x)$ and $Q = \mathrm{D}$ are

$$C_r = (-1)^r B_j^{(r-1)}(x)/r(r-2)! \qquad r \geqslant 2. \tag{12}$$

Since $B_j(x)$ is an odd function of x,

$$B_j^s(0) = 0 = B_j^{(2s)}(0) \qquad s = 1, 2, 3, \ldots. \tag{13}$$

It follows from equations (5) and (13) that $\langle (m/j)^{2p}\rangle_0$ (the subscript '0' denoting the value at $x = 0$) is a sum of products of $B_j^{(2q-1)}(0)$ (i.e. derivatives of $B_j(x)$ and also those of odd orders only) such that each term is homogeneous in $B_j(x)$ and D of degree $2p$ (see equations (15) below). If $B_j^{(2q-1)}(0)$ occurs n_q times in a term contributing to $\langle (m/j)^{2p}\rangle_0$, then the condition for homogeneity is

$$\sum_q 2qn_q = 2p. \tag{14}$$

This is precisely the condition for the common factor j^{2p} occurring in equation (7) to cancel exactly with the j^{2q} coming from the denominators of $B_j^{(2q-1)}(0)$ (see equation (21) below) contributing to $\langle(m/j)^{2p}\rangle_0$.

One can obtain from equation (5) the following:

$$\langle(m/j)^2\rangle_0 = B_j^{(1)}(0) \tag{15a}$$

$$\langle(m/j)^4\rangle_0 = 3(B_j^{(1)}(0))^2 + B_j^{(3)}(0) \tag{15b}$$

$$\langle(m/j)^6\rangle_0 = 15(B_j^{(1)}(0))^3 + 15B_j^{(1)}(0)B_j^{(3)}(0) + B_j^{(5)}(0). \tag{15c}$$

From equations (5), (11)-(14) the coefficient of $(B_j^{(1)}(0))^p$ in the expansion of $\langle(m/j)^{2p}\rangle_0$ is found to be $(2p-1)!! = (2p-1)(2p-3)\ldots\times 3\times 1$ and the coefficient of $B_j^{(2p-1)}(0)$ to be unity (see equations (15)). Note that for the special case of $p=1$ these two coefficients are unity as they should be (see equation (15a)) since $1!!=1$. It is interesting to note that $\mathrm{Tr}(J_\lambda^{2p})$ can be evaluated from the derivatives of the Brillouin function.

2.2. *Expansion of $Tr(J_\lambda^{2p})$ in terms of traces of lower powers of J_λ*

From equations (2) and (6), we also have

$$\mathrm{Tr}(J_\lambda^p) = j^p \lim_{x\to 0} \mathrm{D}^p(Z) \qquad p \geqslant 0. \tag{16}$$

Since Z is an even function of x, $\mathrm{D}^{2p+1}(Z)$ is an odd function of x vanishing at $x=0$. Hence $\mathrm{Tr}(J_\lambda^{2p+1})=0$ and

$$\mathrm{Tr}(J_\lambda^{2p}) = j^{2p} \lim_{x\to 0} \mathrm{D}^{2p-1}[ZB_j(x)] \qquad p\geqslant 1 \tag{17}$$

since $\mathrm{D}(Z) = ZB_j(x)$. Applying Leibnitz' theorem to equation (17) and using equations (13) and (16) we have

$$\mathrm{Tr}(J_\lambda^{2p}) = \sum_{q=1}^{p} \binom{2p-1}{2q-1} \mathrm{Tr}(J_\lambda^{2p-2q})[j^{2q}B_j^{(2q-1)}(0)] \qquad p\geqslant 1. \tag{18}$$

In equation (18) we have made use of the convention that $\mathcal{J}^0 = I$, the unit matrix, for any matrix $\mathcal{J}$. The binomial coefficients are denoted by $\binom{n}{r}$. Thus $\mathrm{Tr}(J_\lambda^{2p})$ can be expanded in terms of $\mathrm{Tr}(J_\lambda^{2r})$, $r=0,1,2,3,\ldots,p-1$; $p\geqslant 1$. By repeatedly using equation (18) we can see that $\mathrm{Tr}(J_\lambda^{2p})$ can be expanded in terms of $B_j^{(2q-1)}(0)$.

It has been proved in I that the trace of a product of angular momentum matrices (given either in a cartesian or a spherical basis) can be expanded in terms of $\mathrm{Tr}(J_\lambda^{2p})$. It is particularly pleasing to note that $\mathrm{Tr}(J_\lambda^{2p})$ itself can now be expanded in terms of traces of lower (even) powers of J_λ via the derivatives of the Brillouin function.

3. Standard properties of $\mathrm{Tr}(J_\lambda^{2p})$

3.1. *$Tr(J_\lambda^{2p})$ as a polynomial in $\eta = j(j+1)$*

Since (Abramowitz and Stegun 1970)

$$\coth(y) = \sum_{n=0}^{\infty} 2^{2n}B_{2n}y^{2n-1}/(2n)! \tag{19}$$

and $(2j+1)^2 = 4\eta+1$, it follows from equations (3) and (19) that

$$B_j(x) = \sum_{q=1}^{\infty} [(4\eta+1)^q - 1]B_{2q}x^{2q-1}/(2q)!j^{2q}. \tag{20}$$

Here B_{2n} are the Bernoulli numbers (Abramowitz and Stegun 1970, Arfken 1970). From equation (20) we get

$$B_j^{(2q-1)}(0)=(2q)^{-1}[(4\eta+1)^q-1]B_{2q}/j^{2q} \qquad q\geqslant 1. \tag{21}$$

From the binomial theorem, we have

$$(4\eta+1)^s-1=4\eta f_{s-1}(\eta) \qquad s\geqslant 1 \tag{22}$$

where $f_{s-1}(\eta)$ is a polynomial in η of degree $(s-1)$ with positive integral coefficients. Incidentally the coefficient of x^{2q-1} in the power series expansion of $B_j(x)$ is j^{-2q} times η times a polynomial in η of degree $q-1$ (with rational coefficients).

It is clear from equations (7), (13), (14), (21), (22) and the discussions following equations (13) and (14) that

$$\mathrm{Tr}(J_\lambda^{2p})=\Omega G_{p-1}(\eta) \qquad p\geqslant 1 \tag{23}$$

where

$$\eta=j(j+1) \qquad \Omega=\eta(2j+1). \tag{24}$$

In equation (23), $G_{p-1}(\eta)$ is a polynomial in η of degree $p-1$ with rational coefficients.

Alternatively, from equations (18) and (21) we have

$$\begin{aligned}&\mathrm{Tr}(J_\lambda^2)=\Omega/3 \qquad \mathrm{Tr}(J_\lambda^4)=(\Omega/15)(3\eta-1)\\&\mathrm{Tr}(J_\lambda^6)=(\Omega/21)(3\eta^2-3\eta+1).\end{aligned} \tag{25}$$

Equation (23) follows easily from equations (18), (21), (22), (24), (25) and induction. This qualitative result was obtained earlier using different mathematical techniques (I, Kaplan and Zia 1979, Rashid 1979).

Using equation (21), we can retrieve our earlier results for $\mathrm{Tr}(J_\lambda^{2p})$ (see I, II and III) from equation (18) which can be regarded as a multi-term recurrence relation for the trace polynomials. It is easy to see that equations (15) are consistent with equations (25).

3.2. *The constant term of $G_{p-1}(\eta)$*

From equations (18), (21), (23) and (24) we have

$$\mathrm{Tr}(J_\lambda^{2p})=(2j+1)j^{2p}B_j^{(2p-1)}(0)+\dots \tag{26}$$

and

$$\eta G_{p-1}(\eta)=(2p)^{-1}[(4\eta+1)^p-1]B_{2p}+\dots. \tag{27}$$

In equation (27), for $p\geqslant 2$, the remaining terms have η^2 as a common factor (see equations (14), (15), (21) and (22)). Since

$$\lim_{\eta\to 0}[(4\eta+1)^p-1]/\eta=4p \qquad p\geqslant 1 \tag{28}$$

it follows from equations (27) and (28) that

$$G_{p-1}(0)=2B_{2p} \qquad p\geqslant 1 \tag{29}$$

as shown in II (see equation (4.8)) using a different method.

3.3. *The common denominator of $Tr(J_\lambda^{2p})$, $p \geqslant 1$, is always odd*

It is seen from equations (25) that for $2r = 2, 4, 6$, the common denominator of $\mathrm{Tr}(J_\lambda^{2r})$ in its lowest terms is odd. We shall prove by induction that this is true in general for $r \geqslant 1$.

As shown in the appendix, the denominator of $[(4\eta+1)^q - 1]\ B_{2q}/2q$, $q \geqslant 1$, is always odd (B_{2q} is a rational number). Since the binomial coefficients occurring in equation (18) are integers, it follows from equations (18), (21), (25) and induction that the common denominator of $\mathrm{Tr}(J_\lambda^{2p})$, $p \geqslant 1$, is always odd (see also the results of I, II and III).

Since (see I and III)

$$2\ \mathrm{Tr}(J_L^{2p-2}J_M^2) = 2\ \mathrm{Tr}(J_L^{2p-2}J_N^2) = \eta\ \mathrm{Tr}(J_L^{2p-2}) - \mathrm{Tr}(J_L^{2p}) \tag{30}$$

where L, M and N denote any permutation of x, y and z (L, M and N are different), the denominator of $\mathrm{Tr}(J_L^{2p-2}J_M^2)$ is always twice an odd integer (see table 1 of I and III). Thus the nature (even or odd) of the denominator of $\mathrm{Tr}(J_\lambda^{2p-2}J_\mu^2)$, $p \geqslant 1$, λ, $\mu = x$ or y or z has been clearly established. This result is very useful in the generation of these type of traces by means of recurrence relations (II and III). Next we give a prescription for finding the denominator of $\mathrm{Tr}(J_\lambda^{2p})$.

Let D_{p-1} be the denominator of $G_{p-1}(\eta)$ in its lowest terms. When $p = 1$, $G_0(\eta)$ has only a constant term ($= 2B_2 = \frac{1}{3}$) and hence $\mathrm{D}_0 = 3$. Since D_{p-1} is proved to be odd, let

$$\mathrm{D}_{p-1} = I_f I_s I_t \qquad p \geqslant 2 \tag{31}$$

where I_f, I_s and I_t are all odd positive integers to be determined as follows.

I_f is a product of (odd) prime numbers such that (a) each prime factor of I_f is a divisor of at least one of those numbers which exceed by 1 a non-trivial divisor of $2p$ (i.e. excluding unity and $2p$ itself); (b) I_f is *quadratfrei* (Hardy and Wright 1960): I_f contains *no* prime factor raised to a power higher than the first. Since 2 is a non-trivial divisor of $2p$ for $p \geqslant 2$, I_f is always divisible by 3. Hence all the denominators D_n, $n \geqslant 0$, are divisible by 3. This result is consistent with the fact that $3G_n(2) = 1$, $n \geqslant 0$ (see equation (4.10) of II).

The integer I_s is given by

$$I_s = \begin{cases} p_k^{\alpha-1} & \text{if } 2p+1 = p_k^\alpha,\ \alpha \geqslant 2,\ p_k \text{ is an odd prime} \\ 2p+1 & \text{otherwise.} \end{cases} \tag{32}$$

Since $2p = p_k^\alpha - 1$, $\alpha \geqslant 2$, has $p_k - 1$ as a non-trivial divisor, I_f has p_k as an odd prime factor. In other words D_{p-1} will not be quadratfrei in this case.

The number I_t is the least positive odd integer such that

$$K = p(2p-1)\mathrm{D}_{p-1}/\mathrm{D}_{p-2} \qquad p \geqslant 2 \tag{33}$$

is an integer, D_{p-2} being the denominator of $G_{p-2}(\eta)$. Obviously K is divisible by the greatest power of 2 which divides p since $2p-1$, D_{p-1} and D_{p-2} are all odd.

The underlying principles for our prescription for the denominators are: (i) the coefficient of η^{p-1} in $G_{p-1}(\eta)$ is $(2p+1)^{-1}$ (see § 4 below); (ii) the constant term of $G_{p-1}(\eta)$ is $2B_{2p}$ (see § 3.2). By the von Staudt-Clausen theorem (Ramanujan 1927, Hardy and Wright 1960, Arfken 1970) the denominator of B_{2q}, $q \geqslant 1$, is the continued product of prime numbers which are the next numbers (in the natural order) to the factors of $2q$ (including unity and $2q$ itself). In other words the denominator of B_{2q},

$q \geqslant 1$ is quadratfrei and that it is twice an odd integer (see also table 1 of Ramanujan (1927) and Abramowitz and Stegun (1970)); (iii) the coefficients $a_i (= N_i/D_{p-1}$, N_i being the numerator of a_i) of $G_{p-1}(\eta) = \Sigma_{i=0}^{p-1} a_i \eta^i$ can be generated by means of recurrence relations starting with either a_{p-1} or a_0 and knowing the coefficients of $G_{p-2}(\eta)$ (see equations (3.3)-(3.5) of II). Thus we have obtained a von Staudt-Clausen-Ramanujan type prescription for the denominator of $G_{p-1}(\eta)$.

For the sake of completeness, we present in table 1 the denominator of $G_{p-1}(\eta)$ for $18 \leqslant 2p \leqslant 52$ along with the corresponding I_f, I_s, I_t and K values (see equations (31)-(33)). The values of $R = 2D_{p-1}/d_{2p}$ are also given, d_{2p} being the denominator of B_{2p}. From the von Staudt-Clausen-Ramanujan theorem, it is clear that $R \geqslant 1$ and is odd. Knowing the denominator of $G_{p-1}(\eta)$ and a_{p-1} we have computed by a program $\mathrm{Tr}(J_\lambda^{2p})$ up to $2p = 52$ using the recurrence relations given in II. We have applied certain checks to $G_{p-1}(\eta)$ as given in II and found our results to be correct. The numerator and the denominator of B_{2p} found from our calculations agree with those given in Ramanujan (1927) and Abramowitz and Stegun (1970). We are thus optimistic that our prescription for D_{p-1} will also work for higher p. It is observed from table 1 that D_{p-1} is quadratfrei whenever $2p+1$ itself is prime (in this case I_f and I_s are quadratfrei). We believe that this is true in general, but we have not proved it.

Table 1. The denominator D_{p-1} of $G_{p-1}(\eta) = \Omega^{-1}\mathrm{Tr}(J_\lambda^{2p})$, the factors I_f, I_s, I_t of D_{p-1} and the quantities K and R. As usual $\eta = j(j+1)$, $\Omega = \eta(2j+1)$, $D_{p-1} = I_f I_s I_t$, $R = 2D_{p-1}/d_{2p}$, d_{2p} being the denominator of the Bernoulli number B_{2p}. $K = p(2p-1)D_{p-1}/D_{p-2}$. Note that D_{p-1} is divisible by 3; it is quadratfrei whenever $2p+1$ is prime. $D_7 = 255$ (see I).

$2p$	D_{p-1}	$I_f/3$	I_s	I_t	K	R
18	1 995	5×7	19	1	1 197	5
20	3 465	5×11	3×7	1	330	21
22	345	1	23	5	23	5
24	6 825	$5\times7\times13$	5	1	5 460	5
26	189	7	3^2	1	9	63
28	435	5	29	1	870	1
30	7 161	7×11	31	1	7 161	1
32	58 905	5×17	3×11	7	4 080	231
34	105	1	5×7	1	1	35
36	959 595	$5\times7\times13\times19$	37	1	5757 570	1
38	4 095	5	3×13	7	3	1 365
40	47 355	$5\times7\times11$	41	1	9 020	7
42	49 665	$5\times7\times11$	43	1	903	55
44	108 675	5×23	$3^2\times5$	7	2 070	315
46	4 935	1	47	5×7	47	35
48	162 435	$5\times7\times13\times17$	7	1	37 128	7
50	21 879	11×13	3×17	1	165	663
52	61 215	5×7	53	11	3 710	77

4. Sum rules for the Bernoulli numbers B_{2q} and the Riemann zeta functions $\zeta(2q)$ and $\zeta(1-2q)$

From equation (18) we now obtain sum rules for the Bernoulli numbers and hence for the Riemann zeta functions.

If a_{p-1} is the coefficient of η^{p-1} in $G_{p-1}(\eta)$ defined by equation (23), then (see equation (4.7) of II)

$$a_{p-1}=(2p+1)^{-1} \qquad p\geqslant 1. \tag{34}$$

Now from equations (18), (21), (23), (24) and (34), we have, after some simple algebra, an interesting sum rule for B_{2q}:

$$\sum_{q=1}^{p}\binom{2p+1}{2q}2^{2q}B_{2q}=2p \qquad p\geqslant 1. \tag{35}$$

In the symbolic notation which replaces the equals sign by the symbol $\doteqdot$ to indicate that the two expressions will be equal when exponents are lowered to subscripts (see, for example, Rainville 1967) equation (35) takes a very simple form:

$$(2B+1)^{2p+1}\doteqdot 0 \qquad p\geqslant 1. \tag{36}$$

If the left-hand side of equation (36) is expanded binomially and $(2B)^r$ is replaced by 2^rB_r, equation (35) is obtained since (Abramowitz and Stegun 1970) $B_0=1$, $B_1=-1/2$, $B_{2k+1}=0$, $k\geqslant 1$. Equation (36) is a special case of a relation given by Lucas (1891).

Equation (18) can be a source of sum rules for B_{2q}. Thus, since (I and III)

$$\mathrm{Tr}(J_\lambda^{2r})=2^{1-2r} \qquad r\geqslant 0 \qquad j=\tfrac{1}{2} \qquad \eta=\tfrac{3}{4} \tag{37a}$$

$$\mathrm{Tr}(J_\lambda^{2r})=2+\delta_{r0} \qquad r\geqslant 0 \qquad j=1 \qquad \eta=2 \tag{37b}$$

it follows from equations (18), (21) and (37) that

$$\sum_{q=1}^{p}\binom{2p}{2q}(4^q-1)2^{2q}B_{2q}=2p \qquad p\geqslant 1 \tag{38}$$

and

$$2\sum_{q=1}^{p}\binom{2p}{2q}(9^q-1)B_{2q}=4p-(9^p-1)B_{2p} \qquad p\geqslant 1. \tag{39}$$

As the Bernoulli numbers are intimately related to the Riemann zeta functions through the relations (Abramowitz and Stegun 1970)

$$\text{(a)} \qquad B_{2n}=(-1)^{n-1}2(2n)!\zeta(2n)/(2\pi)^{2n} \qquad n\geqslant 1 \tag{40}$$

$$\text{(b)} \qquad B_{2n}=-2n\zeta(1-2n) \qquad n\geqslant 1 \tag{41}$$

one can easily obtain from equations (35) and (38)-(41) corresponding sum rules for $\zeta(2q)$ and $\zeta(1-2q)$. Thus, for example,

$$\sum_{q=1}^{p}(-1)^{q-1}\binom{2p+1}{2q}(2q)!\zeta(2q)/\pi^{2q}=p \qquad p\geqslant 1 \tag{42}$$

and

$$p+\sum_{q=1}^{p}\binom{2p}{2q}(4^q-1)q2^{2q}\zeta(1-2q)=0 \qquad p\geqslant 1. \tag{43}$$

Equations (35), (38), (39), (42) and (43) are simple sum rules for the Bernoulli numbers and the Riemann zeta functions. They have been independently checked and found correct for $p\leqslant 11$.

Alternatively equations (35), (42) and (43) (for example) can be regarded as recurrence relations for B_{2q}, $\zeta(2q)$ and $\zeta(1-2q)$ respectively. Although Ramanujan (1927) had obtained many recurrence relations for B_{2q} based on quite different ideas, relations (38) and (39) seemed to have escaped his attention.

5. Conclusion

We have shown that $\mathrm{Tr}(J_\lambda^{2p})$ can be developed from the derivatives (of odd orders) of the Brillouin function and that this trace can be expanded in terms of traces of lower (even) powers of J_λ. A von Staudt-Clausen-Ramanujan type prescription has been given for the denominators of the trace polynomials. Some results concerning these polynomials are only rederived but some are apparently new. As interesting corollaries sum rules and recurrence relations for the Bernoulli numbers and the Riemann zeta functions have been obtained.

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Appendix

In this appendix we prove that the denominator of

$$T = [(4\eta+1)^q - 1]B_{2q}/2q \qquad q \geqslant 1 \tag{A1}$$

in its lowest terms is always odd (B_{2q} is a rational number).

From the fundamental theorem of arithmetic (Hardy and Wright 1960), we have for $q > 1$

$$q = 2^\beta N \qquad \beta \geqslant 0 \qquad N = \text{odd}. \tag{A2}$$

Repeatedly using $a^2 - b^2 = (a+b)(a-b)$, we find, for $\beta > 0$,

$$(4\eta+1)^q - 1 = \left(\prod_{i=1}^{\beta} [(4\eta+1)^{2^{-i}q} + 1]\right)[(4\eta+1)^N - 1]. \tag{A3}$$

Using the binomial theorem we have

$$(4\eta+1)^r + 1 = 2f_r(\eta) \qquad r \geqslant 1 \tag{A4}$$

$$(4\eta+1)^N - 1 = 4\eta f_{N-1}(\eta) \qquad N \geqslant 1 \tag{A5}$$

so that, for $\beta \geqslant 0$,

$$(4\eta+1)^q - 1 = 2^\beta(4\eta)f_{q-1}(\eta) \qquad q \geqslant 1. \tag{A6}$$

In equations (A4)-(A6), $f_s(\eta)$ is a polynomial in η of degree s with positive integral coefficients. Hence from equations (A2) and (A6) the denominator of T is the denominator of $2B_{2q}/N$. Since N is odd (see equation (A2)), it is enough to show that the denominator of $2B_{2q}$ is odd. The von Staudt-Clausen-Ramanujan theorem (Ramanujan 1927, Hardy and Wright 1960, Arfken 1970) implies that the denominator of B_{2q}, $q \geqslant 1$, is always twice an odd integer (see also § 3.3). It is now clear that the denominator of T, in its lowest terms, is always odd for $q \geqslant 1$.

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